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T1 ________________	**60758**	F1 ________________
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2017

Mathematical Contest in Modeling (MCM/ICM) Summary Sheet

A New Multiple Dam System: Guardian Angels of the Zambezi River

Summary

After a 50-year service, the Kariba Dam now is threatened with some security risks. Our goal is to design a better multiple dam system to replace the current dam. The system as we think is more efficient and flexible, so that it will be safer and more adaptable. To be specific, this system can properly handle emergency water flow situations (i.e. flooding and/or prolonged low water conditions). Also, it provides a reasonable balance between safety and costs.

Totally we build 3 models. To simplify the models, we regard the shape of the valley as an inverted trapezoid and we suppose several dams will be built along the river. In model Ⅰ, we calculate the water storage capacity of each dam, and take elements like head as constraints. Finally, we regard the balance between the maximum power generation capacity and costs as the criterion for optimization. According to the risk and return algorithm of the investment in the linear programming, the number and position of the dam are selected. Model Ⅱ is an improved version of model Ⅰ.In this model we regard the water storage model as a changeable one. In model Ⅲ we mainly consider the relationship among the reservoir water level and the flow as well as the time. We introduce Chézy Formula in our model and use it to calculate the velocity of water flow in the river valley, that is to say, we take the time delay of water transmission between reservoirs into consideration.

More factors like evaporation, precipitation, and infiltration need to be analyzed. We would like to focus on the impact of these factors on the reservoir water level and to conduct a sensitivity analysis.

Contents

1 Introduction

1.1 Background

Before we start our work, it is very necessary to take the background information of the Zambezi River and the Kariba Dam into consideration. Zambezi River, as the fourth-longest river in Africa, the longest east-flowing river in Africa and the largest flowing into the Indian Ocean from Africa, plays an important role in provide water for the agriculture and industry as well as some other aspects. The Kariba Dam, standing 128 meters tall and 579 meters long, supplies 40% of electricity to the local area. Also, it protects the local people from the danger of flood and drought.

After a 50-year service for Southern Africa, the Kariba Dam is threatened with some security risks. Spillway torrents have excavated a massive cavern in the Zambezi river bed, now 10 times bigger and deeper than the original design dimensions, that threatens the stability of the wall foundations [1]. If the dam falls, the consequences will be very serious. For example, the water flow may destroy the Cahora Bassa hydroelectric power station. Besides, it will also bring troubles to the lives and property of about 3500000 residents in Zambia, Zimbabwe, Mozambique and Malawi, as well as the power supply in the region. So there is an urgent need for a viable solution to this problem.

1.2 Restatement of the Problem

We are required to remove the Kariba Dam and replace it with a multiple dam system along the Zambezi River. This system, as we think, should work as good or even better than the Kariba Dam. We will give the number and placement of the new dams and justify our solution.

In order to establish such a system, the following requirements needs to be met:

- Our strategy needs to provide a reasonable balance between safety and costs.
- Our new system should be able to handle emergency water flow situations (i.e. flooding and/or prolonged low water conditions).
- Our strategy should provide guidance to the ZRA managers.
- Our strategy should include information addressing any restrictions regarding the locations and lengths of time that different areas of the Zambezi River should be exposed to the most detrimental effects of the extreme conditions.

2 Assumptions

To simplify the problem, we make the following basic assumptions.

- Only the section between Vitoria falls and Chora Bassa Lake is suitable for building dams.

- After the removal of the Kariba Dam, the shape of the valley below the lake surface is similar to the shape of the upper and lower reaches of the river near the Kariba Lake.
- The drop height between the Victoria falls and Chora Bassa Lake varies evenly with the distance.
- The valley is shaped like an inverted isosceles trapezoid.
- When the water level reaches the normal reservoir water level, the reservoir water level no longer changes.
- The natural flow of the first dam is the flow of the Victoria Falls.
- Evaporation, infiltration and Tributary flows will be ignored.

3 Terms and Definitions

Table 1 Terms and Definitions

Symbols	Definitions
P_r	Actual power generation capacity
Q_i	Outflow of water from a dam
H	Net head
u	The efficiency of converting water into electricity
g	Acceleration of gravity
n	Number of new dams
P_i	The power generation capacity of number *i* dam
P_{sum}	the total power generation capacity of n dams
$P_{\max}$	Maximum power generation capacity

l_{0i}	The length of the water surface of reservoir *i*
h_{bi}	The maximum depth between the water surface and the bottom of the dam *i*
h_d	Total head
θ_1	The ideal angle on both sides of the valley
θ_2	The ideal angle of the valley
V_r	The volume of water storage of Lake Kariba
V_i	The volume of water storage of dam *i*
a	The width of the dam's bottom
X_i	The horizontal distance between dam *i* and dam *i*-1
h_{ai}	The vertical distance between the bottom of dam *i* and the bottom of dam *i*-1
$S_i(t)$	The water storage of dam *i* when the time is *t*
$S_i(t-1)$	The water storage of dam *i* when the time is *t-1*
$O_i(t)$	The flow from reservoir *i* to reservoir *i+1*
$I_i(t)$	The flow from reservoir *i-1* to reservoir *i*
S_i^{Max}	The water storage under the design flood level of dam *i*

S_i^{Min}	The water storage under dead water level of dam *i*
v_i	The mean velocity between dam *i-1* and dam *i*
C	The Chézy coefficient
R	The hydraulic radius (~ water depth)
h_i	The water depth between dam *i-1* and dam *i*
ι	The bottom slope
τ_i	The time spent by the flow in the travel from dam *i-1* to dam *i*
η	Manning's roughness coefficient
A_i	The cross-sectional area of the water between dam *i-1* and dam *i*
$p\omega$	The wetted perimeter

4 Model

To explain all the problems clearly, we build 3 models. The first one is about water storage and power generation. But we find it not complete, so we build the second one in which we take one important situation into consideration, namely whether the lower reservoir will reach the upper dam between every 2 adjacent dams. The third model describes the temporal and spatial relationship among natural water flow, reservoir water storage and displacement. Thus we get the required parameters.

4.1 Model Ⅰ

4.1.1 The model Description

The dam and valley will form a reservoir which keeps the water from the upper reaches and the water in the reservoir gradually accumulates until the water level reaches the normal water level. the volume of water storage can be regarded as a three-dimensional graphics. The sum of all reservoir water storage capacity is the total water storage capacity. The power generation capacity is influenced the outflow of the dam, the net head, the efficiency of the generator system and the acceleration of gravity.

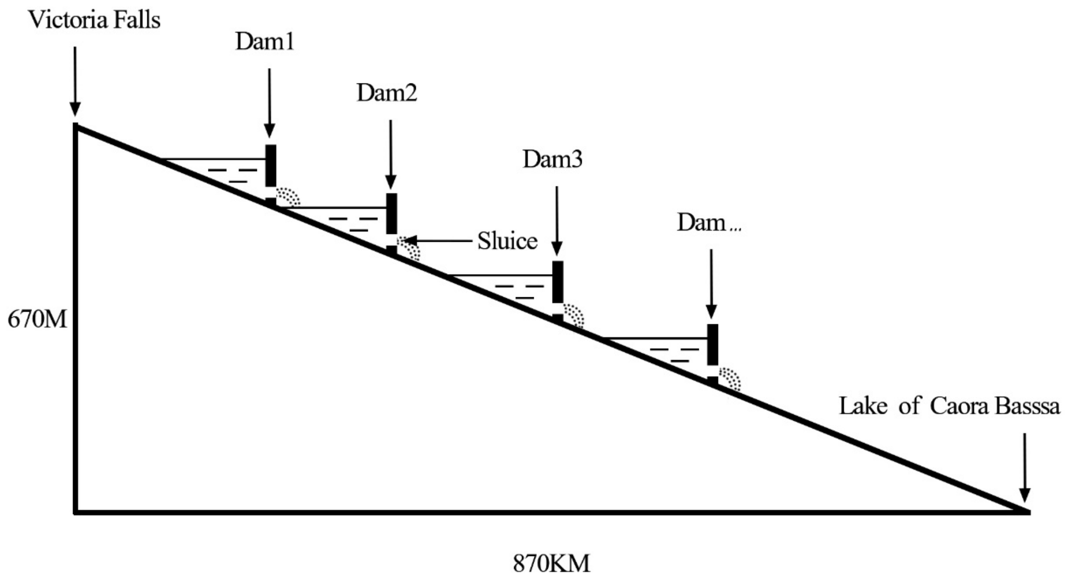

Figure 1 multiple dam system

4.1.2 Additional Assumptions

- When the hydropower station works normally，the inflow is equal to the outflow, and the outflow is equal to the flow through the generator turbine.
- The net head is equal to the height between the reservoir surface and the dam bottom.
- The water level between 2 dams is low enough and can’t reach the bottom of the upper dam.

4.1.3 Symbols

- P_r refers to the actual power generation capacity.
- Q_i refers to the outflow of water from a dam.
- H refers to the net head.
- u refers to the efficiency of converting water into electricity.

- g refers to the acceleration of gravity.
- n refers to the number of new dams.
- P_i refers to the power generation capacity of number *i* dam.
- P_{sum} refers to the total power generation capacity of n dams.
- $P_{\max}$ refers to the maximum power generation capacity.
- l_{0i} refers to the length of the water surface of reservoir *i*.
- h_{bi} refers to the maximum depth between the water surface and the bottom of the dam *i*.
- h_d refers to the total head.
- θ_1 refers to the ideal angle on both sides of the valley.
- θ_2 refers to the ideal angle of the valley.
- V_r refers to the volume of water storage of Lake Kariba.
- V_i refers to the volume of water storage of dam *i*.
- a refers to the width of the dam's bottom.

4.1.4 Model Establishment

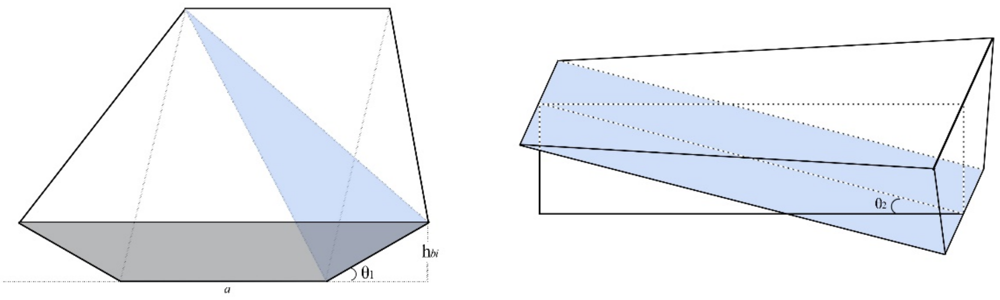

Figure 2 The reservoir model between two adjacent dams

According to the knowledge of geometry, we simplify this model with 2 3-dimension geometric figures, namely a pyramid and a tetrahedron, so that it can be relatively easily calculated.

Then we have:

$$V_i = \frac{(2a+2h_{bi}/\tan\theta_1)h_{bi}\dfrac{h_{bi}}{\tan\theta_2}}{2\times 3} + \frac{ah_{bi}\dfrac{h_{bi}}{\tan\theta_2}}{2\times 3} \qquad (1)$$

To meet the requirement, namely the total volume water storage of n dams should be no more than the volume of water storage of Lake Kariba, we have:

$$\sum_{i=1}^{n} V_i \geq V_r \qquad (2)$$

Then we get the relationship between n and h_b, so we have:

$$\sum_{i=1}^{n} h_{bi} \leq h_d \quad (3) \quad , \quad 10 \leq n \leq 20 \quad (4)$$

According to the relationship between net head and power generation capacity[2], we have:

$$P_i = gQ_iH_iu$$

When $P_{\max}$ is maximum, n and h_b are the optimal solutions, then we have:

$$P_{\max} = Max\left\{\sum_{i=1}^{n} P_i\right\}$$

4.1.5 Solution

According to formula $(1),(2),(3),(4)$ we can get the relationship between variable

$h_b,\ n, V_r$:

$$V_r = \left[\frac{(2a + 2h_b / \tan\theta_1) h_b \dfrac{h_b}{\tan\theta_2}}{2\times 3} + \frac{a h_b \dfrac{h_b}{\tan\theta_2}}{2\times 3} \right] \cdot n \qquad (5)$$

In this model the dams are idealized, the value of h_{bi} is h_b,the values of θ_1, θ_2 and a are collected from Google Earth. Some values of the variables are as follows.

$$a = 400m$$

$$\tan\theta_1 = \frac{1}{5}$$

$$\tan\theta_2 = \frac{670}{870000}$$

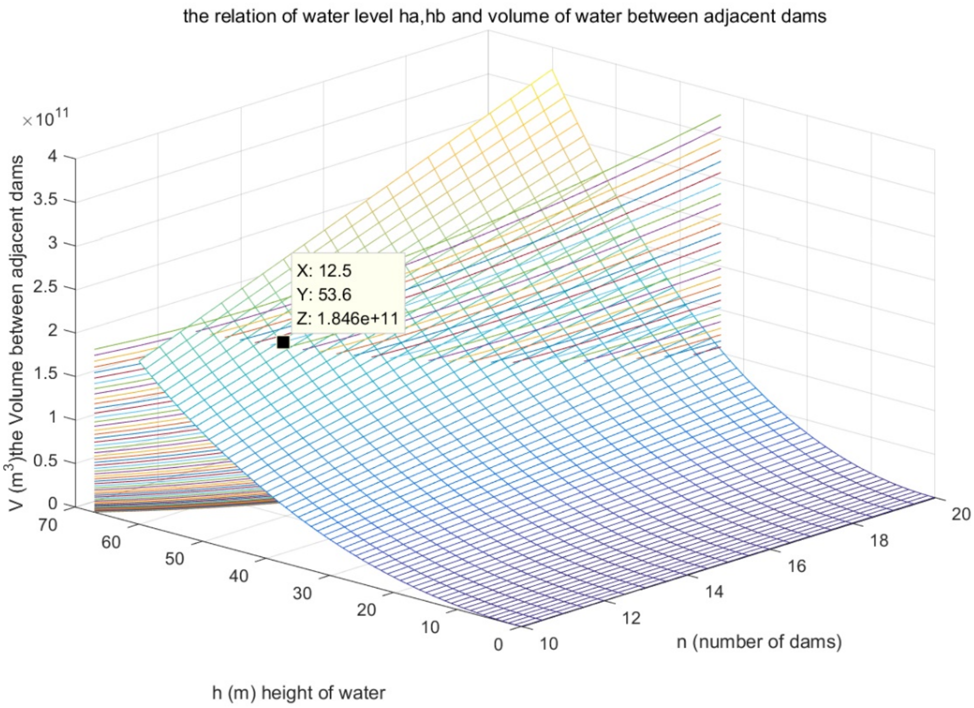

Figure 3 the relation of water level ha,hb and volume of water

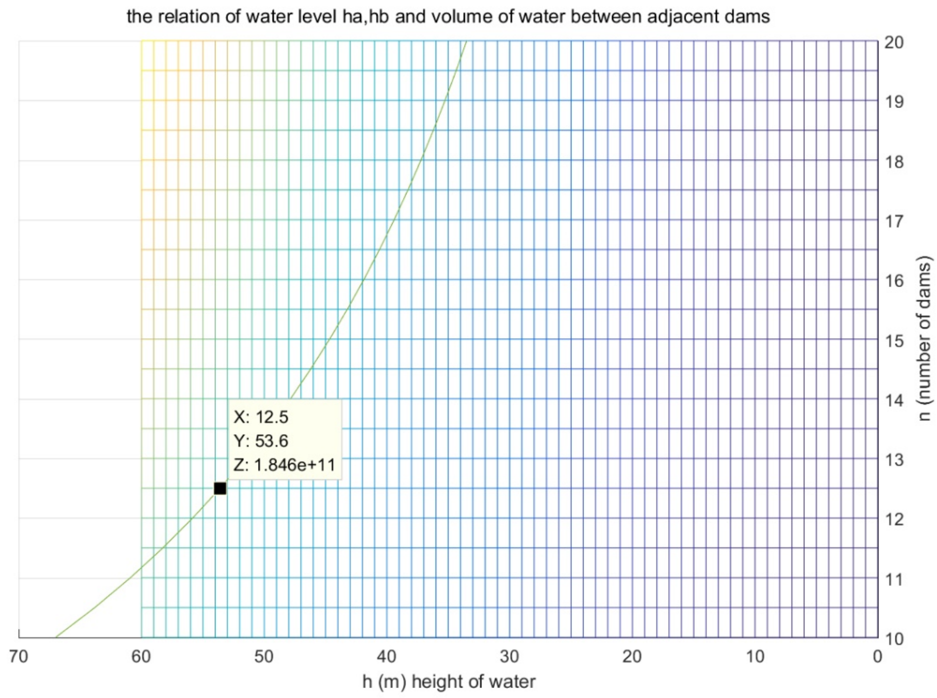

Figure 4 top view of the Three-dimensional graphics

4.1.6 Results and Analysis

We use MATLAB to draw formula (5) and constraint relationship formula (3) in the same coordinate system. As shown in Figure 3, the two surfaces intersect at one line. As shown in Figure 4, The total water storage capacity of the cascade dam should not be less than that of the current dam. Also, to balance costs and efficiency, we can get the Optimal values when $Z = 1.85\times10^{11} m^3$, namely $n = 13, h_b = 54m$. Without other restrictions, in order to make this multiple dam system fully functional, we have chosen to distribute the dam system evenly over the river.

4.1.7 Strengths and Weaknesses

Strengths:

- The reservoir is idealized as geometric figure;
- The optimal value is calculated according to the investment return and risk algorithm criteria.
- The model is simple and convenient to calculate.

Weaknesses:

- In model one we ignore the situation that the lower reservoir may reach the upper dam between every 2 adjacent dams.

- The model is too ideal to have an accurate result.

4.2 Model Ⅱ

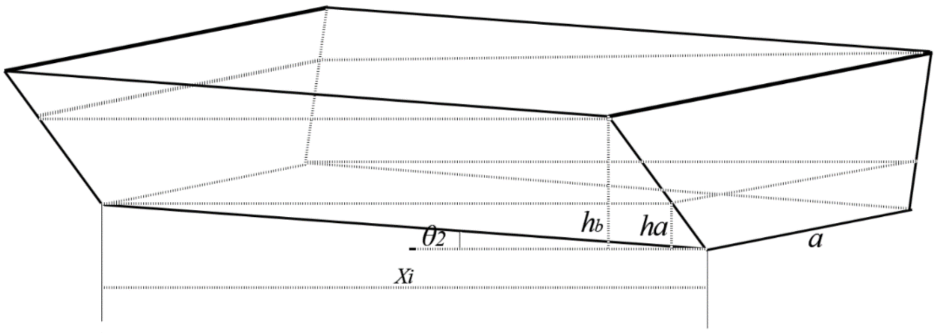

Figure 5 multiple dam system

4.2.1 The Model Description

In model one we assume that the lower reservoir will not reach the upper dam between every 2 adjacent dams. To make our water storage model better, we considered a relatively more general situation.

4.2.2 Additional Assumptions

- The water outlet of upper dam is always higher than or equal to the surface of the reservoir.

4.2.3 Symbols

- X_i refers to the horizontal distance between dam *i* and dam *i*-1.
- h_{ai} refers to the vertical distance between the bottom of dam *i* and the bottom of dam *i*-1.

4.2.4 Model Establishment

According to the shape of the water in the reservoir, we can calculate the volume of water in the reservoir:

$$V_i = \frac{\left(2a + 10h_{ai}\right) \cdot h_{ai} \cdot X_i}{2 \times 3} + \frac{a \cdot X_i \cdot h_{ai}}{2 \times 3} + \frac{X_i}{3}\left[\left(h_{bi} - h_{ai}\right) \cdot \left(2a + 10h_{bi}\right) + \sqrt{\left(a + 5h_{bi} - 5h_{ai}\right) \cdot \left(a + 5h_{bi} + 5h_{ai}\right) \cdot \left(h_{bi} - h_{ai}\right)^2}\right]$$

4.2.5 Solution

According to formula $(1),(2),(3),(4)$, we can get the relationship between variable h_{bi}, h_{ai}, V_i:

$$V_i = \frac{\left(2a + 2\frac{h_{ai}}{\tan\theta_1}\right) \cdot h_{ai}^2}{2 \times 3 \cdot \tan\theta_2} + \frac{a \cdot h_{ai}^2}{2 \times 3 \cdot \tan\theta_2} + \frac{h_{ai}}{3 \cdot \tan\theta_2}\left[\left(h_{bi} - h_{ai}\right) \cdot \left(2a + 2\frac{h_{bi}}{\tan\theta_1}\right) + \sqrt{\left(a + \frac{h_{bi}}{\tan\theta_1} - \frac{h_{ai}}{\tan\theta_1}\right) \cdot \left(a + \frac{h_{bi}}{\tan\theta_1} + \frac{h_{ai}}{\tan\theta_1}\right) \cdot \left(h_{bi} - h_{ai}\right)^2}\right] \tag{6}$$

In this model the values of θ_1, θ_2 and a are collected from Google Earth. Some values of the variables are as follows.

$$a = 400m$$

$$\tan\theta_1 = \frac{1}{5}$$

$$\tan\theta_2 = \frac{670}{870000}$$

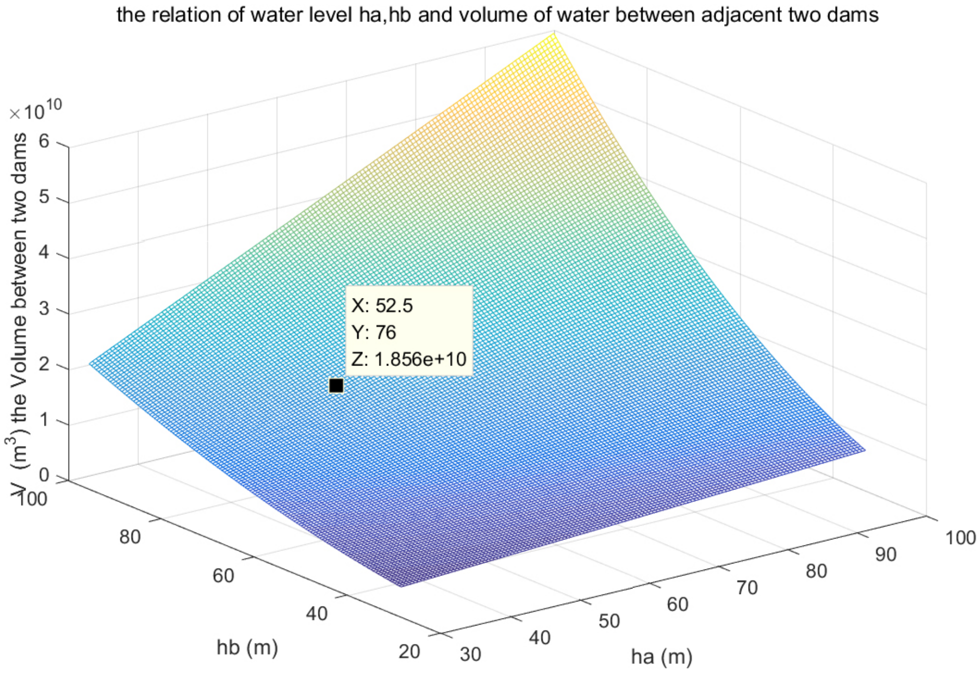

Figure 6 the relation of and the volume of water

4.2.6 Results and Analysis

We use MATLAB to draw formula (6) in the coordinate system. As shown in Figure 6, With the gradual increase of h_a, h_b, the water storage capacity between adjacent dams gradually increases. Because the total storage capacity of cascade dam should not be less than the current storage capacity of the dam, so the greater the value is, the more the storage capacity increases. As the relationship between the size of the dams and costs is positive, the height of the dam is limited. So it requires more constraint relationships to calculate. For example, the constraint relationship between the distance and number of dams, so this model cannot meet the requirements.

4.2.7 Strengths and Weaknesses

Strengths:

- This model is optimized on the basis of Model Ⅰ, the model of water between adjacent dams will change with the increase of the water storage capacity.
- Optimize the mathematical model of the water storage capacity.

Weaknesses:

- We increase the number of variables in this model, while the constraint relationship is not enough to solve a reasonable value.
- In addition, the larger error of variable contributes to the inaccurate result.

4.3 Model Ⅲ

4.3.1 The model Description

To make our multiple dam system handle emergency water flow situations (i.e. flooding and/or prolonged low water conditions), we need the relationship between the reservoir capacity and the outflow and the inflow, and construct a time-dependent equation. Through the calculation of the difference of flow (that is, the net increase of the volume of water), we can get the relationship between water storage capacity and flood to prevent floods. Theoretically, if the upstream water flow has been greater than or equal to the flood flow of water, the water in the reservoirs will eventually overflow the dam, so we also need to calculate the value of the time it takes for water to flow from the flood limit to the design flood level under a condition of a long time and a large flow.

4.3.2 Additional Assumptions

- Regard the valley as an open channel.
- The speed of the flow in the valley is even.
- When floods occur, the rainfall between the dams is not counted in the flow.
- The maximum amount of flow that can be sustained by each section between the dams is equal.

4.3.3 Symbols

- $S_i(t)$ refers to the water storage of dam *i* when the time is *t*.
- $S_i(t-1)$ refers to the water storage of dam *i* when the time is *t-1*.
- $O_i(t)$ refers to the flow from reservoir *i* to reservoir *i+1*.
- $I_i(t)$ refers to the flow from reservoir *i-1* to reservoir *i*.
- S_i^{Max} refers to the water storage under the design flood level of dam *i*

- S_i^{Min} refers to the water storage under dead water level of dam *i*.
- v_i refers to the mean velocity between dam *i-1* and dam *i*.
- C refers to the Chézy coefficient.
- R refers to the hydraulic radius (~ water depth).
- h_i refers to the water depth between dam *i-1* and dam *i*.
- ι refers to the bottom slope.
- τ_i refers to the time spent by the flow in the travel from dam *i-1* to dam *i*.
- η refers to Manning's roughness coefficient.
- A_i refers to the cross-sectional area of the water between dam *i-1* and dam *i*.
- $p\varpi$ refers to the wetted perimeter.

4.3.4 Model Establishment

According to the flow through the dam, we have:

$$S_i(t) = S_i(t-1) - O_i(t) + I_i(t)$$

Suppose the valley is an open channel, $J = \iota$, according to the Chézy formula[3] and Manning Formula[4], we have:

$$v_i = C\sqrt{RJ}\ ,\quad v_i = C\sqrt{R\iota}\ ,\quad C = \frac{1}{n}\sqrt[6]{R}$$

Plus:

$$v_i \times A_i = Q_i$$

According to the relationship between the depth of the river and the cross-sectional area of the water, we have:

$$v_i^{\ 3} = \frac{Q_i\sqrt[3]{R}}{5\eta^2\left(2a + 10h_i\right)}$$

And:

$$\tau_i = \frac{X_i}{v_i}$$

5 Conclusions

The use of the multiple dam system can effectively achieve the purpose of water management, and each water level of the dam is lower. The water level only need to reach 55m, then the river's head can be shared by the 10 dams, and by this way it can reduce the stress and improve the safety of the dams.

Under different climate conditions, the multiple dam system works well.

When drought come, if the reservoir area is too large, evaporation will be large too. In contrast, the exposure area of the multiple dam system is much smaller, so the evaporation will be smaller.

When flood comes, the multiple dam system works effectively, and it can use the water flow delay between dams to obtain a higher storage capacity than the static calculation.

6 Future Work

In the above-mentioned modeling process, we idealized many conditions in our assumption, such as simplified the riverbed shape, assume the velocity of flow be a fixed value, water of the lower reservoir all comes from the upper one, etc. We hope to simulate these factors with a continuous function, and further optimize the model to obtain better results.

So many of water will flow away from the spillway in order to discharge excess water in a short time. it will lead to the waster of water resources. We hope to use long-term optimal mathematical model [5] of rational scheduling to distribute water resources rationally and improve the effectiveness of power.

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A Brief Assessment of the Three Options Listed

After a 50-year service for Southern Africa, the Kariba Dam is threatened with some security risks. Spillway torrents have excavated a massive cavern in the Zambezi river bed, now 10 times bigger and deeper than the original design dimensions, that threatens the stability of the wall foundations. If the dam falls, the consequences will be very serious. For example, the water flow may destroy the Cahora Bassa hydroelectric power station. Besides, it will also bring troubles to the lives and property of about 3.5 million residents in Zambia, Zimbabwe, Mozambique and Malawi, as well as the power supply in the region. So it is high time that we should take actions to change this situation. Now, we will give a brief analysis of the 3 options from different aspects.

Option 1: Repairing the existing Kariba Dam.

Advantages:

- Compared with the other two options, the rehabilitation project requires less money and workload as well as a shorter duration.
- The dam can still work during the period of the rehabilitation project.

disadvantages:

- The dam is defective, the drawbacks will be there if we just repair it. As we know, the designers intended to make the dam impervious to a one-in-ten-thousand-year flood at first, but their calculations were based on only three decades of Zambezi flow data—a period too short to supply a credible forecasting. So we know that it can still not resist the coming flood if we just repair the Kariba Dam.
- It takes a long time to carry out the necessary due diligence and secure the financing for a complex project like this.
- The rehabilitation project cannot fundamentally solve the problem. In 2015, According to IRMSA published Aon Risk Research Report, it says, according to the information of the World Bank's official website, the rehabilitation project will last ten years, and cost about 2.94 billion US dollars which is not include additional money and time. While the cost of Kariba Dam's construction cost is 4.8 billion US dollars. We can see that the cost of rehabilitation project accounted for about sixty-two percent of the construction project's cost.

Option 2: Rebuilding the existing Kariba Dam.

Advantages:

- The known drawbacks can be corrected. That is to say the dam can be redesigned so that it can be more secure and more efficient.

disadvantages:

- The cost will increase and it will surely increase the financial burden of the local governments. If the Kariba Dam is to be rebuilt, huge capital expenditure will be inevitable, and it mainly includes two aspects, namely demolition and reconstruction.
- It takes a long time to carry out the necessary due diligence and secure the financing

for a complex project like this.

- The dam cannot generate power if it is removed, and it means lack of electricity for a long time. at the same time, the Kariba Lake will dry and it cannot prevent people from floods and droughts.

Option 3: Removing the Kariba Dam and replacing it with a series of ten to twenty smaller dams along the Zambezi River.

Advantages:

- Smaller dams are easier to be built, repaired and managed.
- A new multiple dam system means a more flexible scheduling policy. We can take cascade hydro-plant long-term optimization model to achieve of improving resource utilization rate.
- It is possible to achieve the target of solve the flood and drought problem.

disadvantages:

- This option costs the most. and it will surely increase the financial burden of the local governments. If the Kariba Dam is to be rebuilt, huge capital expenditure will be inevitable, and it mainly includes two aspects, namely demolition and reconstruction.
- It takes a long time to carry out the necessary due diligence and secure the financing for a complex project like this.
- The dam cannot generate power if it is removed, and it means lack of electricity for a long time. at the same time, the Kariba Lake will dry and it cannot prevent people from floods and droughts.
- It is hard to decide the number and the placement of the new dams. It requires skilled technicians to conduct field exploring, analysis and calculation the soil components, river width, and slope of the Zambezi River Basin.